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Analytical solution for transient laminar fully developed free convection in vertical concentric annuli

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Abstract—Analytical solutions for transient fully developed natural convection in open-ended vertical concentric annuli are presented. Four fundamental boundary conditions have been investigated and the corresponding fundamental solutions are obtained. These four fundamental boundary conditions are obtained by combining each of the two conditions of having one boundary maintained at uniform heat flux or at uniform wall temperature with each of the conditions that the opposite boundary is kept isothermal at the inlet fluid temperature or adiabatic. An expression for the transient Nusselt number is given for each case. These fundamental solutions may be used to obtain solutions satisfying more general thermal boundary conditions.

INTRODUCTION

UNSTEADY laminar free convection in vertical concentric annuli is of interest in several engineering applications, such as the early stages of melting and in transient heating of insulating air gaps by heat input at the start-up of furnaces. Also, unsteady laminar free convection is likely to find wider use as it could provide the flow mechanism in some types of solar heating and ventilating passive systems. In modern electronic equipment, the vertical circuit boards include heat generating elements, and this situation can be modelled by parallel heated plates with upward flow in the intervening space. Examples of other applications which may be simulated by such a model are the external surface of electric transformers, small domestic mobile winter oil heaters and some types of radiators of hydronic heating systems.

The annular geometry is widely employed in the field of heat exchangers. A typical application is that of gas-cooled nuclear reactors, in which cylindrical fissionable fuel elements are placed axially in vertical coolant channels within the graphite moderator, the cooling gas flowing along the annuli parallel to the fuel elements.

There has been greatly increased interest and research activity in natural convection. Gebhart *et al.* [1] reviewed this research activity. Unsteady developing laminar free convection in vertical parallel plates has been numerically investigated by Joshi [2], Lee *et al.* [3], Yang *et al.* [4] and Kettleborough [5]. Wang [6] has analytically considered the fully developed transient free convection between vertical plates with periodic heat input. Unsteady laminar free convection in a two-dimensional enclosure is solved using the scaling analysis of Patterson and Imberger [7]. Steady developing laminar natural convection in vertical concentric annuli has been studied by El-Shaarawi and Sarhan [8], El-Arabi *et al.* [9] and Oosthuizen and Paul [10]. Many studies [11, 12] provided analytical solutions for steady fully developed free convection flows in vertical annuli. Different thermal configurations are considered in these studies. However, all of them use boundary-layer assumptions, which are applicable at large Rayleigh numbers. The obtained results show that at relatively low Rayleigh number, or sufficiently large height to gap width ratios (l/b), fully developed conditions can be achieved before the fluid reaches the annulis exit cross-section.

Fully developed free convection flows are obtained when the inertia forces vanish and a balance is attained between the pressure and gravitational forces on the one hand and the viscous forces on the other hand. The study of such flows gives the limiting conditions for developing flows and provides an analytical check on numerical solutions. The lack of analytical solutions for transient fully developed laminar natural convection in vertical concentric annuli, with different thermal boundary conditions, motivated the present work. The purpose of this paper is to present, in closed forms, transient fully developed free convection solutions, corresponding to four fundamental thermal boundary conditions, in vertical concentric annuli.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

We consider unsteady laminar fully developed free convection flow inside a vertical concentric annulus of a finite length (1), as shown in Fig. 1, immersed in a stagnant fluid of infinite extent maintained at a constant temperature T_0. The fluid inside the annulus

NOMENCLATURE

b	annular gap width, $r_2 - r_1$
c_p	specific heat of fluid at constant pressure
D	equivalent (hydraulic) diameter of annulus, $2b$
D_w	diameter of heat transfer boundary
f	volumetric flow rate, $\int_{r_1}^{r_2} 2\pi r u \, dr$
F	dimensionless volumetric flow rate, $f/(\pi l \gamma \, Gr^*)$
g	gravitational body force per unit mass
Gr	Grashof number, $\pm g(T_w - T_0)D^3/\gamma^2$ in the case of an isothermal boundary or $\pm gqD^4/2k\gamma^2$ in the case of uniform heat flux (UHF) heat transfer boundary, the plus and minus signs apply to upward (heating) and downward (cooling) flows, respectively. Thus Gr is a positive number in both cases
Gr^*	modified Grashof number, $D\,Gr/l$
h	heat transfer coefficient
k	thermal conductivity of fluid
l	height of annulus
L	dimensionless height of annulus, $1/Gr^*$
N	annulus radius ratio, r_1/r_2
Nu	local Nusselt number, $D(\partial T/\partial r)_w/(T_w - T_0)$
$\overline{Nu}$	average Nusselt number, $\int_0^l Nu \, dz/l$
p	pressure of fluid inside the channel at any cross-section
p'	pressure defect at any point, $p - p_s$
p_0	pressure of fluid at the channel entrance
p_s	hydrostatic pressure, $\rho_0 g z$
P	dimensionless pressure defect at any point, $p' r_2^4/\rho_0 l^2 \gamma^2 Gr^{*2}$
Pr	Prandtl number, $\mu c_p/k$
q	heat flux at the heat transfer surface, $q = \pm k(\partial T/\partial r)_w$ where the minus and plus signs are, respectively, for heating and cooling in case (I). These signs should be reversed in case (O)
r	radial coordinate
r_1	inner radius of annulus
r_2	outer radius of annulus
R	dimensionless radial coordinate, r/r_2
Ra	Rayleigh number, $Gr\,Pr$
Ra^*	modified Rayleigh number, $Gr^*\,Pr$
t	dimensionless time, $\bar{t}k/\rho c r_2^2$
$\bar{t}$	time
T	fluid temperature at any point
T_i	initial fluid temperature
T_m	mixing cup temperature, $\int_{r_1}^{r} ruT \, dr / \int_{r_1}^{r} ru \, dr$
T_0	fluid temperature at channel entrance
T_r	reference temperature
T_w	temperature of the wall
u	axial velocity component at any point
U	dimensionless axial velocity, $ur_2^2/(l\gamma\,Gr^*)$
z	axial coordinate
Z	dimensionless axial coordinate, $z/(l\,Gr^*)$.

Greek symbols

α	constant appearing in equation (7)
β	volumetric coefficient of thermal expansion
β_n	eigenvalues
γ	kinematic viscosity of fluid, μ/ρ_0
θ	dimensionless temperature, $(T - T_0)/(T_w - T_0)$ in the case of an isothermal heat transfer boundary and $(T - T_0)/(qD/2k)$ for UHF boundary
θ_m	dimensionless mixing cup temperature, $(T_m - T_0)/(T_w - T_0)$ in the case of an isothermal heat transfer boundary and $(T_m - T_0)/(qD/2k)$ for UHF boundary
θ_r	dimensionless reference temperature, $(T_r - T_0)/(T_w - T_0)$ in the case of an isothermal heat transfer boundary and $(T_r - T_0)/(qD/2k)$ for UHF boundary
θ_w	dimensionless wall temperature, $(T_w - T_0)/(T_w - T_0) = 1$ in the case of an isothermal heat transfer boundary and $(T_w - T_0)/(qD/2k)$ for UHF boundary
λ_n	eigenvalues
μ	dynamic viscosity of fluid
ρ	fluid density at temperature T, $\rho_0(1 - \beta(T - T_0))$
ρ_0	fluid density at T_0.

initially has the temperature T_0, and suddenly, at least one of the annular walls is heated or cooled so that its temperature is different from the ambient temperature T_0. Due to fully developed flow assumptions the fluid enters the part of the annular passage under consideration with an axial velocity profile which remains invariant in the entire channel (i.e. $\partial u/\partial z = 0$). The fluid is assumed to be Newtonian, enters the channel at the ambient temperature T_0, and has constant physical properties, but obeys the Boussinesq approximation according to which its density is constant except in the gravitational terms of the vertical momentum equation. Viscous dissipation and internal heat generation are absent.

Under the above mentioned assumptions and using the dimensionless parameters given in the Nomenclature, the equations of continuity, motion and energy reduce to the following two simultaneous non-dimen-

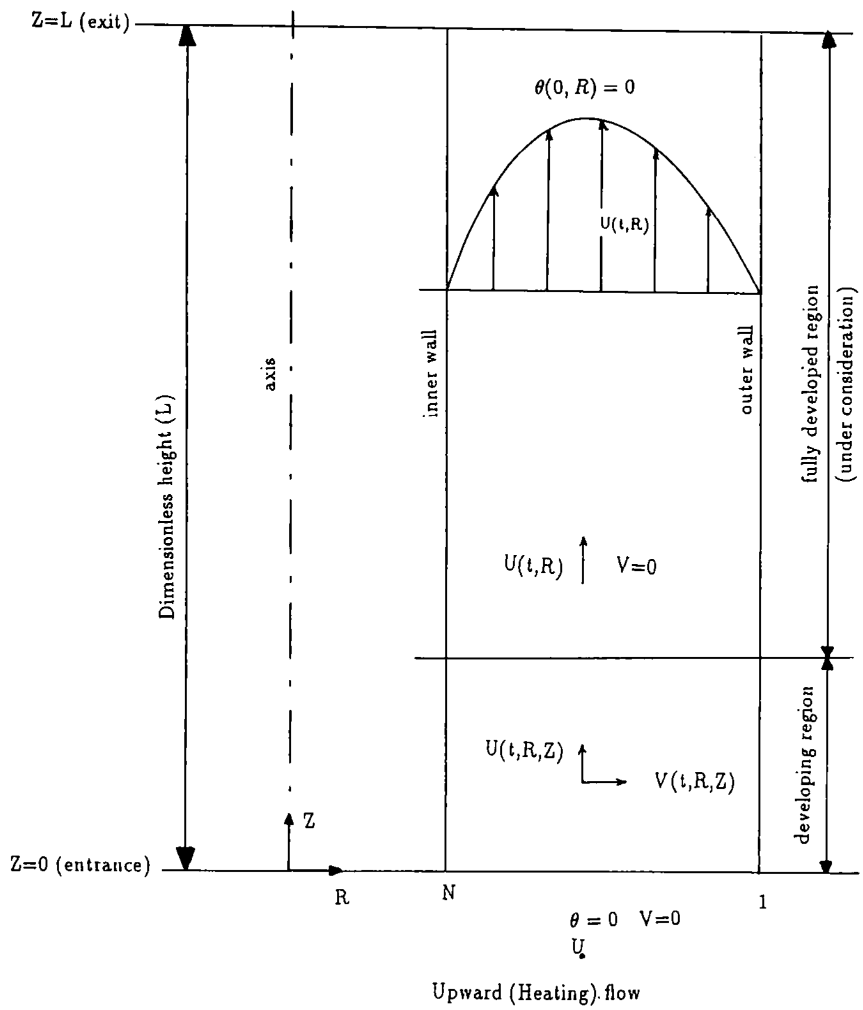

FIG. 1. Schematic diagram.

sional equations:

$$\frac{\partial U}{\partial t} = Pr\frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial U}{\partial R}\right] - Pr\frac{\partial P}{\partial Z} + \frac{Pr}{16(1-N)^4}(\theta-\theta_r) \tag{1}$$

$$\frac{\partial \theta}{\partial t} + U\,Pr\frac{\partial \theta}{\partial Z} = \frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial \theta}{\partial R}\right] \tag{2}$$

where θ_r is a reference temperature which is equal to zero for an open-ended channel and which can be determined from the condition of zero net flow for a closed-ended channel.

Two initial conditions and four boundary conditions are therefore needed to obtain a solution for the above two equations. The two initial conditions are

$$U(0,R) = \theta(0,R,Z) = 0. \tag{3}$$

The two boundary conditions related to U are

$$U(t,1) = U(t,N) = 0. \tag{4}$$

On the other hand, there are many possible thermal boundary conditions applicable to the annular configuration. In the present paper, the non-dimensional parameters used in the formulation of the problem are chosen to suit annuli having their two boundaries at two different heat fluxes (q_1 and q_2) or at two different uniform temperatures (T_1 and T_2) or annuli under one of four fundamental boundary conditions. These four fundamental boundary conditions are obtained by combining each of the two conditions of having one boundary maintained at uniform wall temperature or at specified heat flux with each of the conditions that the opposite boundary is kept isothermal at the inlet fluid temperature (T_0) or adiabatic ($\partial T/\partial r = 0$).

With the two boundaries of an annulus maintained at UHF (uniform heat flux) conditions, if q_1 refers to the higher heat flux then q_1 will be at the hotter wall in the case of heating and at the cooler wall in the case of cooling. Similarly, when the two boundaries of an annulus are kept isothermal, T_1 refers to the wall which has the larger temperature difference from T_0. Thus, T_1 is the temperature of the hotter wall in the case of heating the two boundaries and of the cooler wall in case of cooling both boundaries.

From the previous discussion it may be seen that there are many thermal boundary conditions applicable to the annulus geometry. However, under certain conditions, the energy equation (2) becomes linear and homogeneous in θ (e.g. when $\partial\theta/\partial Z$ is constant), and then any linear combination of solutions will be a solution. It may then be possible to develop certain special or fundamental solutions to this equation satisfying particular or specific boundary conditions, which can be combined to satisfy any other boundary conditions. This method is known as the method of superposition. Reynolds *et al.* [13] defined four fundamental boundary conditions which produce four fundamental solutions to the energy equation (2) when it becomes linear. For the sake of completeness, these fundamental solutions are stated hereinafter.

(1) Fundamental solution of the first kind, which satisfies the boundary conditions of a temperature step change at one wall, the opposite wall being kept isothermal at the inlet fluid temperature. Using the present notation, this corresponds to $\theta = 1$ at one wall and $\theta = 0$ at the opposite wall for $t > 0$, where the boundaries are kept at the inlet fluid temperature, $\theta = 0$ for $t \geqslant 0$ for all cases.

(2) Fundamental solutions of the second kind which satisfy the boundary conditions of a step change in heat flux at one wall, the opposite wall being adiabatic. Using the present notation, this corresponds to $\partial\theta/\partial R = -1/(1-N)$ at the inner wall and $\partial\theta/\partial R = 0$ at the outer wall or $\partial\theta/\partial R = 0$ at the inner wall and $\partial\theta/\partial R = 1/(1-N)$ at the outer wall for $t > 0$.

(3) Fundamental solutions of the third kind which satisfy the boundary conditions of a temperature step change at one wall, the opposite wall being adiabatic. This corresponds to $\theta = 1$ at one wall and $\partial\theta/\partial R = 0$ at the opposite wall for $t > 0$.

(4) Fundamental solutions of the fourth kind where a step change in heat flux at one wall is applied while the opposite wall being kept isothermal at the inlet fluid temperature. This corresponds to $\partial\theta/\partial R = -1/(1-N)$ at the inner wall while $\theta = 0$ at the outer wall or $\theta = 0$ at the inner wall and $\partial\theta/\partial R = 1/(1-N)$ at the outer wall for $t > 0$.

With any of the above mentioned boundary conditions, the boundary opposite to that maintained adiabatic (i.e. $\partial\theta/\partial R = 0$) or isothermal (i.e. $\theta = 0$) is termed the heat transfer boundary (even though there is transfer of heat through a boundary maintained at $\theta = 0$). For each of the above fundamental solutions, two cases are considered, namely, case (I), in which the heat transfer boundary is at the inner wall and case (O) in which the heat transfer boundary is at the outer wall. The aim of the present paper is to obtain the above mentioned four fundamental solutions.

GENERAL ANALYSIS

Substituting θ from equation (1) into equation (2), we obtain

$$\frac{\partial^2 U}{\partial t^2} - \frac{Pr}{R}\frac{\partial}{\partial R}\left(R\frac{\partial^2 U}{\partial t\,\partial R}\right) + Pr\frac{\partial^2 P}{\partial t\,\partial Z} + Pr^2 U\frac{\partial^2 P}{\partial Z^2} = \frac{1}{R}\frac{\partial}{\partial R}\left\{R\left[\frac{\partial^2 U}{\partial t\,\partial R} - Pr\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right)\right)\right]\right\}. \tag{5}$$

A solution of equation (5) in the form $U = U(t, R)$ is only possible if

$$\frac{\partial^2 P}{\partial Z\,\partial t} = \gamma(t) \tag{6}$$

and

$$\frac{\partial^2 P}{\partial Z^2} = \alpha(t). \tag{7}$$

Integrating equation (6) with respect to time yields

$$\frac{\partial P}{\partial Z} = \beta(t) + F(Z) \tag{8}$$

from which

$$\frac{\partial^2 P}{\partial Z^2} = F'(Z) = \alpha(t). \tag{9}$$

But $F'(Z)$ is independent of time, and as a result

$$\frac{\partial^2 P}{\partial Z^2} = \alpha \tag{10}$$

where α is constant. Equation (10) gives the solution for P as

$$P = 0.5\alpha Z^2 + \beta Z + \delta(t). \tag{11}$$

Applying the conditions, for an open-ended channel, that $P = 0$ at both inlet and exit (i.e. at $Z = 0$ and L), gives

$$P = 0.5\alpha Z(Z-L). \tag{12}$$

From equation (1) we have

$$\frac{\partial\theta}{\partial Z} = 16\alpha(1-N)^4 \tag{13}$$

which means that, for a given R in a given annulus, the dimensionless temperature θ varies linearly with the axial distance Z. This implies that the assumption of a hydrodynamically fully developed free convection flow should necessarily mean that the flow is also thermally fully developed, regardless of the value of the Prandtl number (Pr). In other words, for free convection flows in a vertical annulus, the thermal development length is shorter than or at most equal to that of the hydrodynamic development length, irrespective of the value of the Prandtl number. However, in pure forced convection flows, such a result is only obtained if $Pr \leqslant 1$.

As a result of the conclusion that α is constant, equation (5) is reduced to

$$\frac{\partial^2 U}{\partial t^2}-\frac{Pr}{R}\frac{\partial}{\partial R}\left(R\frac{\partial^2 U}{\partial t\,\partial R}\right)+\alpha Pr^2 U$$
$$=\frac{1}{R}\frac{\partial}{\partial R}\left\{R\left[\frac{\partial^2 U}{\partial t\,\partial R}-Pr\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial U}{\partial R}\right)\right)\right]\right\}. \quad (14)$$

The governing equations (1)–(2) can be simplified if one of the two annulus boundaries is kept isothermal. In order to satisfy this boundary condition, θ must, in this particular case, be independent of Z. Thus, it is concluded that α must, in such a case, equal zero. Therefore, equations (12) and (13) reduce, in this case, to the following equations, respectively;

$$P=0 \quad (15)$$

$$\frac{\partial \theta}{\partial Z}=0. \quad (16)$$

For a closed-ended annulus which has at least one of its boundaries at constant wall temperature, integrating equation (6) once, yields

$$\frac{\partial P}{\partial Z}=\beta(t) \quad (17)$$

and for this case the term $Pr\,(\partial P/\partial Z)$, in equation (1), can be combined with θ_r.

Equation (16) states that, in a case with an isothermal boundary, the fully developed temperature profile is a function of R and t only. On the other hand, equation (15) states that the fully developed pressure inside an open-ended annulus of an isothermal boundary is equal to the hydrostatic pressure, at the same elevation, outside the annulus. This implies that, in such a fully developed case with an isothermal boundary, there would be no pressure drop due to fluid viscous drag since this latter is just offset by the buoyancy driving force. If the two governing equations (1) and (2) are solved for the velocity and temperature profiles (U and θ) then the following useful parameters can be evaluated. The dimensionless volumetric flow rate (F) can be evaluated from the following equation:

$$F=2\int_N^1 RU\,\mathrm{d}R. \quad (18)$$

Since for a fully developed flow U is a function of R and t only, it follows that the definite integral on the right-hand side of equation (18) and hence F are functions of t only regardless of the value of the axial coordinate Z, i.e. they are not related to the value of the annulus height. The dimensionless mixing cup temperature is given by

$$\theta_m=\int_N^1 RU\theta\,\mathrm{d}R\bigg/\int_N^1 RU\,\mathrm{d}R. \quad (19)$$

To find the variation of θ_m, in the fully developed flow region, with the dimensionless axial distance Z, the above equation is differentiated with respect to Z. Since U is independent of Z, this gives

$$\frac{\partial \theta_m}{\partial Z}=\int_N^1 RU\frac{\partial \theta}{\partial Z}\mathrm{d}R\bigg/\int_N^1 RU\,\mathrm{d}R, \quad (20)$$

which, on substituting for $\partial\theta/\partial Z$ from equation (13) into the above equation, yields

$$\frac{\partial \theta_m}{\partial Z}=16\alpha(1-N)^4. \quad (21)$$

Integrating equation (21) with respect to Z between annulus entrance and exit, taking into consideration that $\theta_m=0$ at $Z=0$, results in

$$\theta_m=16\alpha(1-N)^4 Z. \quad (22)$$

Using the dimensionless parameters given in the Nomenclature, the following expressions for the local Nusselt number can easily be obtained: For a UWT

$$Nu=\pm 2(1-N)\left(\frac{\partial\theta}{\partial R}\right)_w \quad (23)$$

and for a UHF boundary

$$Nu=\pm\frac{2(1-N)}{\theta_w}\left(\frac{\partial\theta}{\partial R}\right)_w=\frac{2}{\theta_w} \quad (24)$$

where the minus and plus signs apply respectively for cases (I) and (O) when there is heating and vice versa when there is cooling.

From equation (1) it can be seen that $(\partial\theta/\partial R)$ is a function of R and T only which is dependent on the fully developed axial velocity profile (U), i.e. it is independent of Z. Hence, for a case with a UWT boundary, equation (23) shows that the fully developed local Nusselt number is a function of time only. Consequently, the fully developed average Nusselt number is, in this case (UWT), independent of annulus height L. On the other hand, for boundary conditions other than (UWT), provided that the flow is hydrodynamically fully developed, equation (13) shows that the temperature varies linearly with Z. Hence, equation (24) shows that the fully developed local Nusselt number, for this case, varies hyperbolically with Z.

It is important to mention here that, in order to maintain the validity of the hydrodynamic fully developed flow assumption, the only thermal boundary conditions accepted, other than the UWT boundary conditions, are those that vary linearly with Z. As a result, all the problems which include boundary conditions other than the UWT can be considered as fundamental problems of the second kind.

FUNDAMENTAL SOLUTIONS

If at least one of the two annulus boundaries is kept isothermal, equations (1) and (2) are reduced to

$$\frac{\partial U}{\partial t} = Pr\frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial U}{\partial R}\right] + \frac{Pr}{16(1-N)^4}(\theta-\theta_r) \quad (25)$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial \theta}{\partial R}\right]. \quad (26)$$

Equation (26) assumes a solution in the form

$$\theta(t,R) = \theta_1(t,R)+\theta_2(R) \quad (27)$$

where $\theta_2(R)$ accounts for the nonhomogeneousity in the boundary conditions. The solution of the homogeneous part is obtained by the separation of variables as

$$\theta_1(t,R) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t}[Y_0(\lambda_n R)-C_n J_0(\lambda_n R)] \quad (28)$$

and

$$A_n = \frac{\int_N^1 -\theta_2(R)R[Y_0(\lambda_n R)-C_n J_0(\lambda_n R)]\,dR}{\int_N^1 R[Y_0(\lambda_n R)-C_n J_0(\lambda_n R)]^2\,dR} \quad (29)$$

where $\theta_2(R)$, C_n and λ_n depend on the kind of fundamental case we have. Now, equation (25) assumes a solution in the form

$$U(t,R) = U_h(t,R)+U_{p1}(t,R)+U_{p2}(R)+U_{p3}(R) \quad (30)$$

where U_h is the solution of

$$\frac{\partial U_h}{\partial t} - \frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial U_h}{\partial R}\right] = 0 \quad (31)$$

which has the solution

$$U_h(t,R) = \sum_{n=1}^{\infty} D_n e^{-\beta_n^2 t}[Y_0(\beta_n R)-B_n J_0(\beta_n R)] \quad (32)$$

where β_n's are the roots of

$$Y_0(\beta_n)-B_n J_0(\beta_n) = 0 \quad (33)$$

$B_n = Y_0(N\beta_n)/J_0(N\beta_n)$, and

$$D_n = \frac{\int_N^1 -R[U_{p1}(0,R)+U_{p2}(R)+U_{p3}(R)][Y_0(\beta_n R)-B_n J_0(\beta_n R)]\,dR}{\int_N^1 R[Y_0(\beta_n R)-B_n J_0(\beta_n R)]^2\,dR}. \quad (34)$$

U_{p1} is the solution of

$$\frac{\partial U_{p1}}{\partial t} - \frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial U_{p1}}{\partial R}\right] = \frac{\theta_1(t,R)}{16(1-N)^4}. \quad (35)$$

The solution of equation (35) can be obtained by the variation of parameters method as

$$U_{p1}(t,R) = \sum_{n=1}^{\infty}[C_1 J_0(\lambda_n R) + C_2 Y_0(\lambda_n R)+F(R)]\,e^{-\lambda_n^2 t} \quad (36)$$

where

$$\begin{aligned}F(R) &= 0.25\pi\lambda_n A_n R^2 J_0(\lambda_n R)\{[Y_1^2(\lambda_n R)+Y_0^2(\lambda_n R)] \\ &\quad - E_n[J_1(\lambda_n R)Y_1(\lambda_n R)+J_0(\lambda_n R)Y_0(\lambda_n R)]\} \\ &\quad +0.25\pi\lambda_n A_n R^2 Y_0(\lambda_n R)\{-[J_1(\lambda_n R)Y_1(\lambda_n R) \\ &\quad +J_0(\lambda_n R)Y_0(\lambda_n R)]+E_n[J_1^2(\lambda_n R)+J_0^2(\lambda_n R)]\}\end{aligned} \quad (37)$$

where $E_n = Y_0(\lambda_n)/J_0(\lambda_n)$ and the constants C_1 and C_2 are given as

$$C_i = \Delta_i/\Delta \quad i = 1,2 \quad (38)$$

where

$$\Delta_1 = -F(N)Y_0(\lambda_n)+F(1)Y_0(N\lambda_n) \quad (39)$$

$$\Delta_2 = -F(1)J_0(N\lambda_n)+F(N)J_0(\lambda_n) \quad (40)$$

and $\Delta = J_0(N\lambda_n)Y_0(\lambda_n)-J_0(\lambda_n)Y_0(N\lambda_n)$.

It is worth mentioning that we add the homogeneous solution of equation (35) to the particular solution in order to force U_{p1} to satisfy the boundary conditions. Now, U_{p2} is the solution of

$$\frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial U_{p2}}{\partial R}\right] = -\theta_2(R) \quad (41)$$

and as a result, U_{p2} depends on the fundamental case associated with the given thermal boundary conditions. U_{p3} is the solution of

$$\frac{1}{R}\frac{\partial}{\partial R}\left[R\frac{\partial U_{p3}}{\partial R}\right] = \frac{\theta_r}{16(1-N)^4} \quad (42)$$

which has the solution

$$U_{p3}(R) = \frac{\theta_r}{16(1-N)^4}\left[(R^2-1)+(1-N^2)\frac{\ln R}{\ln N}\right]. \quad (43)$$

The determination of $\theta_2(R)$, C_n, λ_n, and U_{p2} depends on the kind of thermal boundary condition and this will be the subject of the following sections.

FUNDAMENTAL SOLUTIONS OF THE FIRST KIND

In this case, the two boundaries of the annulus are kept isothermal, one of which is at the inlet ambient fluid temperature $T_0(\theta = 0)$ while the opposite boundary is at a higher or a lower temperature. The following thermal boundary conditions can be applied:

Case (I): temperature step at the inner wall while the outer wall is kept at the ambient temperature, i.e.

$$\theta(t,1) = 0, \quad \theta(t,N) = 1, \quad t > 0.$$

Case (O): temperature step at the outer wall while the inner wall is kept at the ambient temperature, i.e.

$$\theta(t,N) = 0,\quad \theta(t,1) = 1,\quad t > 0.$$

The evaluation of the required parameters is as follows:

Case (I): the eigenvalues λ_n are the roots of

$$Y_0(N\lambda_n) - C_n J_0(N\lambda_n) = 0 \tag{44}$$

where $C_n = Y_0(\lambda_n)/J_0(\lambda_n)$, and

$$\theta_2(R) = \frac{\ln R}{\ln N}. \tag{45}$$

A_n can be evaluated from equation (29) as

$$A_n = \frac{1}{\lambda_n^2 M \ln N}[\lambda_n R \ln R(C_n J_1(\lambda_n R) - Y_1(\lambda_n R)) + (C_n J_0(\lambda_n R) - Y_0(\lambda_n R))]_N^1 \tag{46}$$

where

$$\begin{aligned} M = \{&0.5R^2[\lambda_n^2 Y_1^2(\lambda_n R) + Y_0^2(\lambda_n R)] \\ &+ 0.5C_n^2 R^2[\lambda_n^2 J_1^2(\lambda_n R) + J_0^2(\lambda_n R)] \\ &- C_n R^2[\lambda_n^2 J_1(\lambda_n R) Y_1(\lambda_n R) \\ &+ J_0(\lambda_n R) Y_0(\lambda_n R)]\}_N^1. \end{aligned} \tag{47}$$

Substituting for $\theta_2(R)$ in equation (41), we get

$$U_{p2}(R) = -\frac{R^2}{4\ln N}[\ln R - 1] + B_1 \ln R + B_2 \tag{48}$$

where $B_1 = [N^2(\ln N - 1) + 1]/(4(\ln N)^2)$ and $B_2 = -1/(4\ln N)$. Closed form expressions for the volume flow rate F and for the non-dimensional mixing cup temperature θ_m, defined by equations (18) and (19), are not possible. However, these parameters can be evaluated numerically. It may be worth mentioning that, in the present case of isothermal boundaries, the temperature θ (and hence θ_m) does not vary with axial distance Z. Thus $\bar{\theta}_m = \theta_m$ This means that the heat transferred to/from the fluid through the two boundaries of the annulus, under the fully developed flow conditions, does not affect the fluid bulk temperature since they are equal and opposite (in order that fully developed conditions can be achieved in such a case). Expressions for the fully developed Nusselt number (local and also average) are obtained after getting the temperature gradient at the walls from equation (27) and then substituting in equation (23). The value of Nu on the inner wall is given as

$$Nu_i(t) = \pm 2(1-N)\left[\left\{\sum_{n=1}^{\infty} \lambda_n A_n e^{-\lambda_n^2 t}[-Y_1(\lambda_n N) + C_n J_1(\lambda_n N)]\right\} + \frac{1}{N\ln N}\right] \tag{49}$$

where the minus and plus signs apply respectively for heating and cooling.

Case (O): the eigenvalues λ_n are still given by equation (44) and

$$\theta_2(R) = 1 - \frac{\ln R}{\ln N} \tag{50}$$

$$\begin{aligned} A_n = \frac{1}{\lambda_n^2 M \ln N}&[\lambda_n R \ln N(C_n J_1(\lambda_n R) \\ &- Y_1(\lambda_n R)) + (\lambda_n R \ln R Y_1(\lambda_n R) + Y_0(\lambda_n R)) \\ &- C_n(\lambda_n R \ln R J_1(\lambda_n R) + J_0(\lambda_n R))]_N^1 \end{aligned} \tag{51}$$

where M is given by equation (47). Substituting for $\theta_2(R)$ in equation (41), we get

$$U_{p2}(R) = \frac{R^2}{4\ln N}[\ln R - (1 + \ln N)] + B_1 \ln R + B_2 \tag{52}$$

where $B_1 = [N^2 - 1 - \ln N]/(4(\ln N)^2)$ and $B_2 = (1 + \ln N)/(4\ln N)$. The expression for Nu on the outer wall is given as

$$Nu_o(t) = \pm 2(1-N)\left[\left\{\sum_{n=1}^{\infty} \lambda_n A_n e^{-\lambda_n^2 t}[-Y_1(\lambda_n) + C_n J_1(\lambda_n)]\right\} - \frac{1}{\ln N}\right]. \tag{53}$$

A sample of the results is plotted in Figs. 2–4. These figures represent the thermal and the hydrodynamics steady state behavior of the fluid for both (I) and (O) cases. As is clear from equations (45), (48)–(50), (52) and (53), the steady state behavior of the annulus is the same as that predicted in ref. [12].

FUNDAMENTAL SOLUTIONS OF THE SECOND KIND

In this case, one of the annulus boundaries is maintained at a constant heat flux (q) and the opposite boundary is perfectly insulated. The governing equations in such a case are equations (1) and (2) where $\partial\theta/\partial Z \neq 0$. We are unable to get a closed form solution for this case.

FUNDAMENTAL SOLUTIONS OF THE THIRD KIND

In this case, since one of the boundaries is isothermal, equations (25) and (26) are the governing equations subject to the following boundary conditions:

Case (I): temperature step at the inner wall while the outer wall is kept insulated, i.e.

$$\frac{\partial\theta(t,1)}{\partial R} = 0,\quad \theta(t,N) = 1\quad t > 0. \tag{54}$$

Case (O): temperature step at the outer wall while the inner wall is kept insulated, i.e.

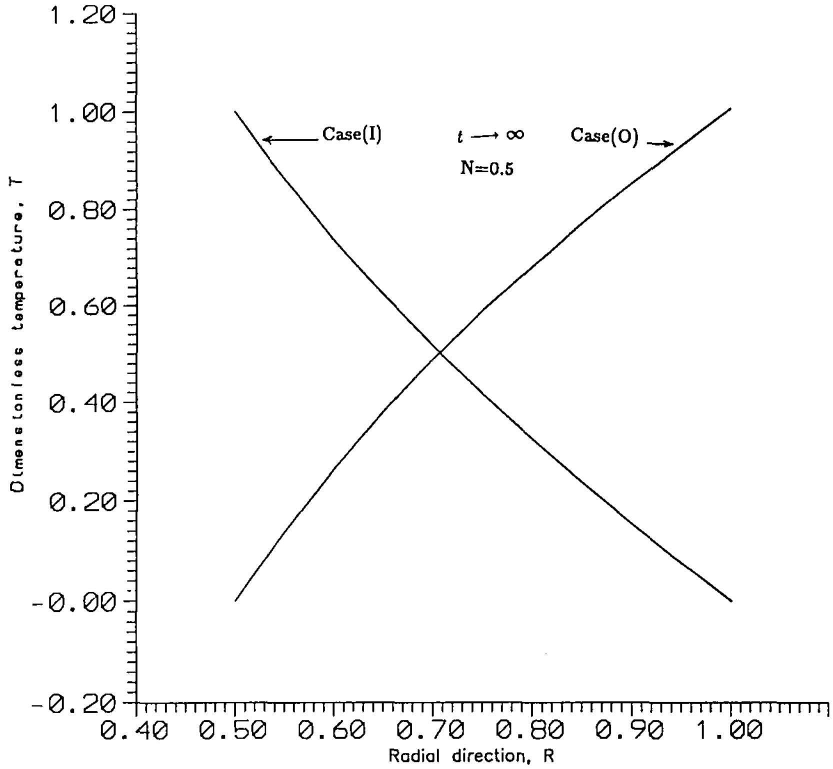

FIG. 2. The dimensionless temperature distribution in the radial direction.

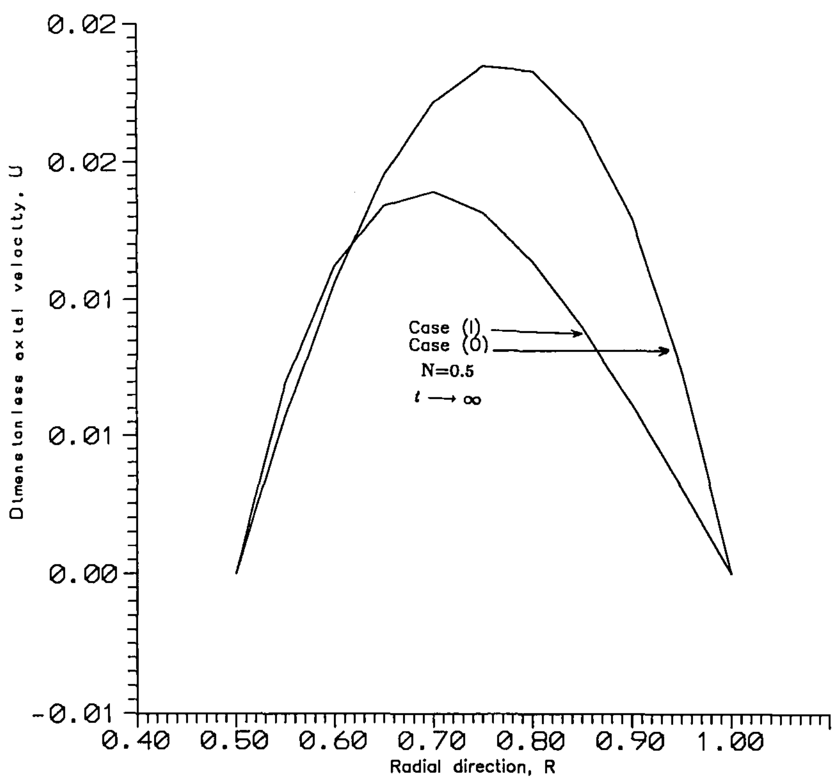

FIG. 3. The dimensionless axial velocity distribution in the radial direction.

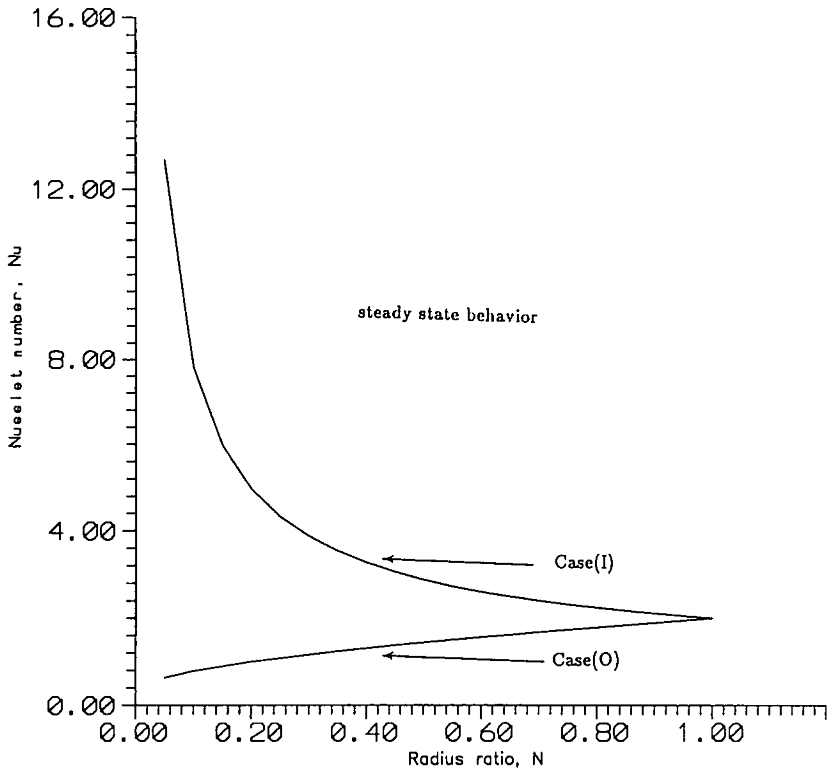

FIG. 4. The variation of Nu with the inner to outer radius ratio N.

$$\frac{\partial\theta(t,N)}{\partial R} = 0, \quad \theta(t,1) = 1 \quad t > 0. \tag{55}$$

The solutions obtained are as follows

Case (I): the eigenvalues λ_n are the roots of

$$Y_1(\lambda_n) - C_n J_1(\lambda_n) = 0 \tag{56}$$

where $C_n = Y_0(N\lambda_n)/J_0(N\lambda_n)$ and

$$\theta_2(R) = 1 \tag{57}$$

$$A_n = \frac{1}{\lambda_n M}[R(C_n J_1(\lambda_n R) - Y_1(\lambda_n R))]_N^1 \tag{58}$$

where M is given by equation (47). The expression for $U_{p2}(R)$ is given as

$$U_{p2}(R) = -\frac{R^2}{4} + B_1 \ln R + B_2 \tag{59}$$

where $B_1 = [N^2-1]/(4\ln N)$ and $B_2 = 1/4$. The local Nusselt number on the inner wall is given as

$$Nu_I(t) = \pm 2(1-N)\left\{\sum_{n=1}^{\infty} \lambda_n A_n \mathrm{e}^{-\lambda_n^2 t}[-Y_1(\lambda_n N) + C_n J_1(\lambda_n N)]\right\}. \tag{60}$$

Case (O): the eigenvalues λ_n are the roots of

$$Y_1(N\lambda_n) - C_n J_1(N\lambda_n) = 0 \tag{61}$$

where $C_n = Y_0(\lambda_n)/J_0(\lambda_n)$, and

$$\theta_2(R) = 1. \tag{62}$$

A_n is given by equation (58) but with the new value of C_n. Also, U_{p2} is given by equation (59). The local Nusselt number on the outer wall is given as

$$Nu_O(t) = \pm 2(1-N)\left\{\sum_{n=1}^{\infty} \lambda_n A_n \mathrm{e}^{-\lambda_n^2 t}[-Y_1(\lambda_n) + C_n J_1(\lambda_n)]\right\}. \tag{63}$$

Note that in both cases (I) and (O) the value of Nu on the insulated wall is zero.

FUNDAMENTAL SOLUTIONS OF THE FOURTH KIND

In this case, since one of the boundaries is isothermal, equations (25) and (26) are the governing equations subject to the following boundary conditions:

Case (I): step change in heat flux at the inner wall while the outer wall is isothermal at the inlet fluid temperature, i.e.

$$\frac{\partial\theta(t,N)}{\partial R} = -1/(1-N), \quad \theta(t,1) = 0 \quad t > 0. \tag{64}$$

Case (O): step change in heat flux at the outer wall while the inner wall is isothermal at the inlet fluid temperature, i.e.

$$\frac{\partial \theta(t,1)}{\partial R} = 1/(1-N), \quad \theta(t,N) = 0 \quad t > 0. \quad (65)$$

The solutions for both cases are given as:

Case (I): the eigenvalues λ_n are the roots of

$$Y_1(N\lambda_n) - C_n J_1(N\lambda_n) = 0 \quad (66)$$

where $C_n = Y_0(\lambda_n)/J_0(\lambda_n)$, and

$$\theta_2(R) = -\frac{N \ln R}{1-N} \quad (67)$$

$$A_n = \frac{N}{\lambda_n^2 M(N-1)}[\lambda_n R \ln R(C_n J_1(\lambda_n R) - Y_1(\lambda_n R)) + (C_n J_0(\lambda_n R) - Y_0(\lambda_n R))]_N^1 \quad (68)$$

$$U_{p2}(R) = -\frac{-N \ln N}{1-N}\left\{\frac{R^2}{4 \ln N}(1-\ln R) + B_1 \ln R + B_2\right\} \quad (69)$$

where

$$B_1 = [1 + N^2(\ln N - 1)]/(4(\ln N)^2)$$

and

$$B_2 = -1/(4 \ln N).$$

The value of Nu on the inner wall is given as

$$Nu_I(t) = \frac{2}{\theta_1(t,N) + \theta_2(N)} \quad (70)$$

where $\theta_1(t, N)$ and $\theta_2(N)$ are given by equations (28) and (67), respectively.

Case (O): the eigenvalues λ_n are the roots of

$$Y_1(\lambda_n) - C_n J_1(\lambda_n) = 0 \quad (71)$$

where $C_n = Y_0(N\lambda_n)/J_0(N\lambda_n)$ and

$$\theta_2(R) = \frac{\ln \frac{R}{N}}{1-N} \quad (72)$$

$$A_n = \frac{1}{\lambda_n^2 M(1-N)}[\lambda_n R \ln N(-C_n J_1(\lambda_n R) + Y_1(\lambda_n R)) - \{\lambda_n R \ln R(-C_n J_1(\lambda_n R) + Y_1(\lambda_n R)) + (-C_n J_0(\lambda_n R) + Y_0(\lambda_n R))\}]_N^1 \quad (73)$$

$$U_{p2}(R) = \frac{1}{4(1-N)\ln N}\{\ln N(1+\ln N)(R^2-1) + (1-N^2-\ln N(R^2-1))\ln R\} \quad (74)$$

and Nu_O on the outer wall is given as

$$Nu_O(t) = \frac{2}{\theta_1(t,1) + \theta_2(1)} \quad (75)$$

where $\theta_1(t, 1)$ and $\theta_2(1)$ are given by equations (28) and (72), respectively.

CONCLUSIONS

Analytical solutions for transient fully developed upward (heating) or downward (cooling) natural convection velocity and temperature profiles in open-ended vertical concentric annuli have been obtained. These solutions correspond to four fundamental boundary conditions obtained by combining each of the two conditions of having one boundary maintained at UHF or at UWT with each of the conditions that the opposite boundary is kept adiabatic or isothermal at the inlet fluid temperature. Expressions for the local Nusselt number are presented for each considered case. Such fully developed values are approached, in a given annulus, when the Rayleigh number (Ra) attains a considerably low value or when the height to gap width ratio (l/b) is sufficiently large. These values represent the limiting conditions and provide analytical checks on numerical solutions for transient developing flows.

Once a developing natural convection flow reaches a state of full development in a given annulus, the volumetric flow rate reaches its upper value; any further increase in the annulus height would not produce an increase in the volumetric flow rate. Moreover, for cases with an isothermal boundary, in a given annulus, the Nusselt number and the mixing cup temperature remain constant spacewise, but vary with time, irrespective of any further increase in the annulus height. However, for cases with two UHF boundary conditions, in a given annulus, we are unable to get a closed form solution.

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